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Algebraic Properties of Subdivision Operators with Matrix Mask and Their Applications

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Subdivision operators play an important role in wavelet analysis. This paper studies the algebraic properties of subdivision operators with matrix mask, especially their action on polynomial sequences and on some of their invariant subspaces. As an application, we characterize, under a mild condition, the approximation order provided by refinable vectors in terms of the eigenvalues and eigenvectors of polynomial sequences of the associated subdivision operator. Moreover, some necessary conditions, in terms of nondegeneracy and simplicity of eigenvalues of a matrix related to the subdivision operator for the refinable vector to be smooth are given. The main results are new even in the scalar case.

Key Words: subdivision operator; transition operator; mask; refinable vector; shift-invariant space; approximation order; accuracy; linear independence.

1. INTRODUCTION

We denote by $l(\mathbb{Z})$ the linear space of all complex valued sequences and by $l_0(\mathbb{Z})$ the linear space of all finitely supported sequences respectively. For r sequences $\lambda_k = (\lambda_k(j))_{j \in \mathbb{Z}} \in l(\mathbb{Z})$, $1 \leqslant k \leqslant r$, we identify $\lambda := (\lambda_1, ..., \lambda_r)^{\mathrm{T}} \in (l(\mathbb{Z}))^r$ with a sequence of components being $r \times 1$ vectors by letting $\lambda(j) = (\lambda_1(j), ..., \lambda_r(j))^{\mathrm{T}}$, $j \in \mathbb{Z}$. Therefore we adopt the notation $\lambda = (\lambda(j))_{j \in \mathbb{Z}}$.

Let $a = (a(j))_{j \in \mathbb{Z}}$ be a finitely supported sequence of matrices, i.e., $a(j)$ are $r \times r$ matrices, all but finite of which are zero. Associated with a there are two operators. One is the *subdivision operator* on $(l(\mathbb{Z}))^r$ defined by

$$S_a \lambda(j) = \sum_{k \in \mathbb{Z}} a^{\mathrm{T}}(j-2k)\, \lambda(k), \qquad j \in \mathbb{Z},$$

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where A^{T} denotes the transpose of a matrix A. The other is the *transition operator* on $(l_0(\mathbb{Z}))^r$ which is defined by

$$T_a \lambda(k) = \sum_{j \in \mathbb{Z}} a(2k - j)\, \lambda(j), \qquad k \in \mathbb{Z}.$$

Subdivision operators and transition operators have been studied extensively. They play an important role in wavelet analysis. The reader is referred to [1, 5, 7] and the references therein for the scalar case, i.e., $r = 1$, and [10] for the vector case. For example, one may use the iteration S_a^n, $n > 0$, of S_a to solve the following equation:

$$\phi = \sum_{j \in \mathbb{Z}} a(j)\, \phi(2 \cdot - j). \tag{1}$$

Equation (1) is called a *refinement equation* with *mask a*. The nonzero solution, $\phi = (\phi_1, ..., \phi_r)^{\mathrm{T}}$, of Eq. (1) is called a *refinable vector*.

The Fourier–Laplace transform of an integrable and compactly supported function f on $\mathbb{C}$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)\, e^{-ix\omega}\, dx, \qquad i = \sqrt{-1}, \quad \omega \in \mathbb{C}.$$

The Fourier–Laplace transform of a compactly supported distribution f is defined by duality. With this notation we may rewrite (1) as

$$\hat{\phi}(\omega) = H(\omega/2)\, \hat{\phi}(\omega/2), \qquad \omega \in \mathbb{C}, \tag{2}$$

where $\hat{\phi}(\omega) = (\hat{\phi}_1(\omega), ..., (\hat{\phi}_r(\omega))^{\mathrm{T}}$ and $H(\omega) = \sum_{j \in \mathbb{Z}} a(j)\, e^{-i\omega}/2$. It is well known that if 1 is an eigenvalue of $H(0)$ and the spectral of $H(0)$ is less than 2, then there exists a compactly supported distribution vector ϕ which satisfies Eq. (1) and $\hat{\phi}(0)$ is an (right) eigenvector of $H(0)$ corresponding eigenvalue 1 [3].

Let $\psi = (\psi_1, ..., \psi_r)^{\mathrm{T}}$ be a compactly supported distribution vector and $b = (b(j))_{j \in \mathbb{Z}} \in (l(\mathbb{Z}))^r$. The semi-convolution of ψ with b, denoted by $\psi *' b$, is defined by

$$\psi *' b = \sum_{j \in \mathbb{Z}} b^{\mathrm{T}}(j)\, \psi(\cdot - j).$$

It is a member of distribution space $\mathscr{D}'(\mathbb{R})$. We denote by $\mathscr{S}(\psi)$ the space of all such semi-convolutions. It is called the *shift-invariant space* generated by ψ. Writing $b(j) = (b_m(j))_{m=1}^r$ we may represent $\psi *' b$ as

$$\psi *' b = \sum_{j \in \mathbb{Z}} \sum_{1 \leqslant m \leqslant r} b_m(j)\, \psi_m(\cdot - j).$$

For any compactly supported distribution vector ϕ, not necessarily refinable, let $K(\phi)=\{\lambda\in(l(\mathbb{Z}))^r : \phi *' \lambda=0\}$. If $K(\phi)=\{0\}$, then the shifts of ϕ are said to be *linearly independent*. Closely related to $K(\phi)$ is the subset of $\mathbb{C}$

$$N(\phi) := \{\theta\in\mathbb{C}\backslash\{0\} : vf_\theta\in K(\phi) \text{ for some } v\in\mathbb{C}^r\backslash\{0\}\},$$

where $f_\theta\in l(\mathbb{Z})$, $\theta\neq 0$, is the exponential sequence $f_\theta(j)=\theta^j$, $j\in\mathbb{Z}$. Recall that (from [9]) $K(\phi)=\{0\}$ if and only if $N(\phi)$ is empty and $\dim K(\phi)<\infty$ if and only if $N(\phi)$ is finite. Moreover, if $K(\phi)$ is of finite dimension, it has a very simple structure. For stating it we introduce some concepts. An $r\times 1$ vector is called a *polynomial vector* if all of its components are algebraic polynomials. Let Π_k^r be the set of all polynomial vectors of degree at most k and Π^r the space of all polynomial vectors, i.e., $\Pi^r=\bigcup_{k\geqslant 0}\Pi_k^r$. If $p\in\Pi^r$, then the sequence $(p(j))_{j\in\mathbb{Z}}$ is called a *polynomial sequence*, which is denoted by p again whenever no confusion may occur. In this sense we can use the same notation Π^r (Π_k^r, resp.) to denote the set of all sequences $(p(j))_{j\in\mathbb{Z}}$, $p\in\Pi^r$ (Π_k^r, resp.).

When $\dim K(\phi)<\infty$, based on a study of certain systems of linear difference equations, Jia and Micchelli gave a characterization of $K(\phi)$ in terms of $N(\phi)$ and polynomial sequences [9]:

$$K(\phi)=\Big\{f : f(j)=\sum_{\theta\in N(\phi)}\theta^j p_\theta(j),\ j\in\mathbb{Z},\ p_\theta \text{ are some polynomial sequences}\Big\}.$$

This characterization plays an important role in our study. Recently the author proved in [2] that $N(\phi)$ is a finite set if and only if $N(\phi)\neq\mathbb{C}$. It is also equivalent to the fact that there exists no sequence $\lambda\in(l_0(\mathbb{Z}))^r$ such that $\phi *' \lambda=0$. Moreover we gave in [2] some necessary and sufficient conditions in terms of $H(\omega)$ that characterize linear independence and stability of the shifts of refinable vectors (see also [4] and [17]). Another approach to deal with the problem of stability is based on the study on eigenvalues of the transition operators, see, e.g., [12].

If $1\notin N(\phi)$ and $\phi\in(L_1(\mathbb{R}))^r$ then $H(0)$ satisfies the following conditions (see [10]): $H(0)$ has 1 as its simple eigenvalue and other eigenvalues of $H(0)$ are less than 1 in modulus. Therefore, in this case, without loss of generality we may assume that $H(0)$ has the form

$$H(0)=\begin{pmatrix}1 & 0\\ 0 & \Lambda\end{pmatrix}, \qquad \lim_{n\to\infty}\Lambda^n=0. \tag{3}$$

For a subset G of $L_p(\mathbb{R})$ and a function $f \in L_p(\mathbb{R})$, the distance of f from G is defined by

$$\operatorname{dist}_p(f, G) := \inf_{g \in G} \|f - g\|_p.$$

Let $\psi \in (L_p(\mathbb{R}))^r$, and $\mathcal{S}^h(\psi) = \{g(\cdot/h)\colon g \in \mathcal{S}(\psi)\}$. We say the space $\mathcal{S}(\psi)$ provides an $L_p(\mathbb{R})$-*approximation of order* s, if for each sufficiently smooth function $f \in L_p(\mathbb{R})$,

$$\operatorname{dist}_p(f, \mathcal{S}^h(\psi)) \leqslant Ch^s, \qquad h > 0,$$

where C is a constant dependent of f, but independent of h.

In [6] Jia proved that for $\psi \in (L_p(\mathbb{R}))^r$, $\mathcal{S}(\psi)$ provides $L_p(\mathbb{R})$-approximation of an integer order k, $k \geqslant 0$, if and only if $\mathcal{S}(\psi)$ has *accuracy* k, i.e., $\Pi_{k-1} \subseteq \mathcal{S}(\psi)$, where, as usual, Π_{k-1} denotes the set of all polynomials of degree at most $k-1$ for $k > 0$ and $\Pi_{-1} = \{0\}$. Note that the concept of accuracy applies to compactly supported distribution vectors as well as to vectors in $(L_p(\mathbb{R}))^r$. For a refinable vector ϕ with refinement mask a, Jia, Riemenschneider, and Zhou [11] characterized the accuracy of $\mathcal{S}(\phi)$ in terms of the behavior of the subdivision operator S_a on some sequences in $(l(\mathbb{Z}))^r$. Among the interesting results proved in [11], the following one is remarkable.

THEOREM JRZ [11]. *For a refinable vector ϕ with mask a, $\mathcal{S}(\phi)$ has accuracy k if and only if there exists a polynomial sequence $q \notin K(\phi)$ such that $S_a q - 2^{-k+1} q \in K(\phi)$.*

It was also proved in [11] that S_a has eigenvalues $1, 2^{-1}, ..., 2^{-k+1}$ provided $\mathcal{S}(\phi)$ has accuracy k. These results extended the work of [16] and [13] by removing the hypothesis about linear independence of the shifts of ϕ.

The paper is organized as follows. In Section 2 we investigate the behavior of subdivision operators on polynomial vectors and on the subspace $K(\phi)$. We shall establish some results. Using these results we characterize in Section 3, under the assumption that $-1 \notin N(\phi)$, the accuracy of $\mathcal{S}(\phi)$ in terms of the eigenvalues and eigenvectors in Π^r. It is worth to point out that the problem of finding nonzero eigenvalues and the associated eigenvectors of polynomial sequences of the subdivision operator had been reduced to an analogous problem corresponding to a matrix of finite order [11]. We shall explain it precisely in Section 4. In Section 4 we discuss the relation between the smoothness of refinable vectors on one hand and the nondegeneracy and simplicity of eigenvalues of some matrices on the other. These matrices are closely related to the transition operators and subdivision operators. Incidentally we prove that smoothness of ϕ implies the accuracy of $\mathcal{S}(\phi)$. Many papers had got such implications. For scalar case, see [1] and [8] etc. For vector case see [14]. Our method is different from any existent one.

2. PROPERTIES OF SUBDIVISION OPERATORS ON POLYNOMIAL SEQUENCES

We begin our study with the behavior of subdivision operators on the polynomial sequences.

LEMMA 1. *For any polynomial vector p we have $S_a p(2j) = q_e(2j)$, $S_a p(2j+1) = q_o(2j+1)$, $j \in \mathbb{Z}$, where q_e and q_o are polynomial vectors given by*

$$q_e(x) = \sum_{k \in \mathbb{Z}} a^{\mathrm{T}}(2k)\, p((x-2k)/2)$$

and

$$q_o(x) = \sum_{k \in \mathbb{Z}} a^{\mathrm{T}}(2k+1)\, p((x-2k-1)/2)$$

respectively.

Proof. By the definition of S_a we have

$$\begin{aligned} S_a p(2j) &= \sum_{k \in \mathbb{Z}} a^{\mathrm{T}}(2j-2k)\, p(k) \\ &= \sum_{k \in \mathbb{Z}} a^{\mathrm{T}}(2k)\, p(j-k) \\ &= q_e(2j). \end{aligned}$$

Similarly, $S_a p(2j+1) = q_o(2j+1)$. ∎

Lemma 1 tells us that both of $S_a p(2\cdot)$ and $S_a p(2\cdot+1)$ are polynomial sequences. We want to know under what conditions $S_a p$ itself is a polynomial sequence. In other words, $q_o = q_e$. To this end, for a polynomial sequence p, we denote by $p(D)$, $D = d/dx$, the differential operator vector induced by p.

THEOREM 1. *Let $F(\omega) = H(\omega/2)$ and p be a polynomial sequence. Then $S_a p$ is a polynomial sequence if and only if the following equality holds*

$$\sum_{k \in \mathbb{Z}} (-1)^k a^{\mathrm{T}}(k)\, p(x-k/2) = 0, \qquad x \in \mathbb{R}, \tag{4}$$

which is equivalent to the fact that for any odd integer n

$$(p^{\mathrm{T}}(x-iD)\,F)(2n\pi)=0, \qquad x\in\mathbb{R}. \tag{5}$$

Furthermore, if (4) (*or* (5)) *is true, then* $q=S_a p$ *is given by*

$$q(x)=\tfrac{1}{2}\sum_{k\in\mathbb{Z}} a^{\mathrm{T}}(k)\,p((x-k)/2). \tag{6}$$

Proof. The proof is similar to that of [1]. We include it for the convenience of the reader. Observe that $S_a p$ is a polynomial sequence if and only if $q_o(2x)-q_e(2x)=0$ for $x\in\mathbb{R}$, which is equivalent to (4). To prove the equivalence of (4) and (5), we see that for any polynomial vector p and odd integer n it holds

$$(p^{\mathrm{T}}(x-iD)\,F)(2n\pi)=\tfrac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k\,p^{\mathrm{T}}(x-k/2)\,a(k), \qquad x\in\mathbb{R}.$$

When (4) (or (5)) holds, equality (6) follows from $S_a p=(q_e+q_o)/2$. ∎

Let p be a polynomial vector. The smallest linear subspace containing all translations of p is defined by

$$ST_p=\operatorname{span}\{p(\cdot-y)\colon y\in\mathbb{R}\}.$$

It is easy to observe that

$$ST_p=\operatorname{span}\{p^{(m)}\colon m\geqslant 0\}, \tag{7}$$

where $p^{(m)}$ is the vector with components being the mth derivatives of those of p. Obviously, ST_p is of finite dimension.

If $S_a\lambda=c\lambda$ for some constant c and $\lambda\in(l(\mathbb{Z}))^r\backslash\{0\}$, c is called an eigenvalue and λ an eigenvector associated with c of S_a.

COROLLARY 1. *For any subdivision operator* S_a *the following conditions hold.*

(i) *Let c be a complex number. Then* $S_a p=cp$ *for* $p\in\Pi_0^r$ *if and only if* $H^{\mathrm{T}}(0)\,p=cp$ *and* $H^{\mathrm{T}}(\pi)\,p=0$, *where, in the last two equations, p is regarded as a vector* $p\in\mathbb{C}^r$.

(ii) *If* $S_a p$ *is a polynomial sequence for some polynomial sequence p, then for any integer* $m\geqslant 0$, $S_a p^{(m)}$ *is a polynomial sequence*;

(iii) *Let* $p\in\Pi^r$ *be an eigenvector of* S_a *associated with eigenvalue c. Then* $S_a p^{(m)}=2^m c p^{(m)}$ *for any integer* $m\geqslant 0$.

Proof. (i) follows from (5) and (6) easily.

If $S_a p$ is a polynomial sequence for some $p \in \Pi^r$, (5) is true by Theorem 1. Consequently, (5) holds by replacing p with $p(\cdot - y)$ for any $y \in \mathbb{R}$. This in turn tells us that $S_a u$ is a polynomial sequence for any polynomial vector $u \in ST_p$ by Theorem 1. Therefore (ii) follows from (7).

Moreover, if $p \in \Pi^r$ is an eigenvector of S_a associated with eigenvalue c. Then, as we have known, $S_a p^{(m)}$ is polynomial sequence for any $m \geqslant 0$. Furthermore by differentiating the two sides of equality

$$cp(x) = \tfrac{1}{2} \sum_{k \in \mathbb{Z}} a^{\mathrm{T}}(k)\, p((x-k)/2)$$

and appealing to (6) in Theorem 1 we obtain (iii).

Remark 1. For $r=1$ and multidimensional case, some results similar to (ii) and (iii) of Corollary 1 were proved in [1].

THEOREM 2. *Assume that $H(0)$ has form* (3). *The following statements are true.*

(i) *If $S_a p = 1/2^k p$ for some nonnegative integer k and $p \in \Pi^r \backslash \{0\}$, then p can be represented as $p = d\varepsilon_1 x^k + \sum_{m<k} p_m x^m$, $p_m \in \mathbb{C}^r$, where d is a constant and $\varepsilon_1 = (1, 0, ..., 0)^{\mathrm{T}} \in \mathbb{R}^r$.*

(ii) *If $p \in \Pi^r$ is an eigenvector of S_a associated with the eigenvalue c such that $p^{(k)} \neq 0$ and $|c| = 1/2^k$, then $c = 1/2^k$.*

Proof. We first conclude $p^{(m)} = 0$ for any $m > k$. For otherwise, there exists some $m > k$ such that $p^{(m)} \in \Pi_0^r \backslash \{0\}$. By (iii) and (i) of Corollary 1 we have $H^{\mathrm{T}}(0)\, p^{(m)} = 2^{-k+m} p^{(m)}$. Thus $H(0)$ has an eigenvalue 2^{-k+m}, which is larger than 1, contradicting the assumption about $H(0)$.

Since $p^{(k)} \in \Pi_0^r$ satisfies $S_a p^{(k)} = p^{(k)}$, it must be a multiple of ε_1 by (i) of Corollary 1. This proves assertion (i).

If $|c2^k| = 1$ and $S_a p = cp$, $p \in \Pi_k^r$ by (i). Moreover $p^{(k)} \in \Pi_k^r \backslash \{0\}$ is an eigenvector of $H(0)$ with eigenvalue $c2^k$ by (i) and (iii) of Corollary 1. Again by the assumption for $H(0)$, $c = 1/2^k$. The proof is complete. ▮

Now we concentrate our attention on the behavior of S_a concerned with $K(\phi)$ and Π^r. We observe that, for a refinable vector ϕ, the following equality holds

$$\sum_{j \in \mathbb{Z}} \lambda^{\mathrm{T}}(j)\, \phi(\cdot - j) = \sum_{j \in \mathbb{Z}} (S_a \lambda)^{\mathrm{T}}(j)\, \phi(2 \cdot - j). \tag{8}$$

Thus we know that $K(\phi)$ is an invariant subspace of S_a.

The following result is a key tool for characterizing the accuracy of $\mathscr{S}(\phi)$ in terms of the eigenvetors of S_a in Π^r. The present proof, which is easier than that in the original version of the manuscript, is provided by one of referees.

THEOREM 3. *Assume that* $-1 \notin N(\phi)$ *and* p *is a polynomial sequence. If* $S_a p - q \in K(\phi)$ *for some polynomial sequence* q, *then* $S_a p$ *is also a polynomial sequence.*

Proof. Let $\Theta \subseteq \mathbb{C} \backslash \{0\}$ be a finite set and q_θ, $\theta \in \Theta$, be some polynomials. We define sequences g_θ, $\theta \in \Theta$, by $g_\theta(j) = \theta^j q_\theta(j)$, $j \in \mathbb{Z}$. It is well known that the finite set of $r \times 1$ vectors g_θ, $\theta \in \Theta$, are linearly independent provided $q_\theta \neq 0$ for any $\theta \in \Theta$.

Let $f := S_a p - q \in K(\phi)$. Since $-1 \notin N(\phi)$, we have by Lemma 2 of [2] that $N(\phi) \neq \mathbb{C}$. Therefore $f(j) = \sum_{\theta \in N(\phi)} \theta^j p_\theta(j)$, $j \in \mathbb{Z}$, where p_θ are some polynomial sequences. On the other hand we have by Lemma 1

$$S_a p(j) = (q_e(j) + q_o(j))/2 + (-1)^j (q_e(j) - q_o(j))/2, \qquad j \in \mathbb{Z}.$$

Noting $-1 \notin N(\phi)$ and applying the above result about linear independence we conclude $q_e - q_o = 0$, as desired. ∎

From Theorem 3 and the fact that $K(\phi)$ is an invariant subspace of S_a it holds

COROLLARY 2. *Assume* $-1 \notin N(\phi)$. *Then* $K(\phi) \cap \Pi^r$ *is an invariant subspace of* S_a.

The condition that "$-1 \notin N(\phi)$" in Theorem 3 cannot be replaced by the condition that "$\theta \notin N(\phi)$ for some $\theta \in \mathbb{C}$," or, equivalently, "$N(\phi) \neq \mathbb{C}$." We shall give a counterexample in the next section.

We note that (cf. [2]) the sufficient conditions for $-1 \notin N(\phi)$ are

(i) $N(\phi) \neq \mathbb{C}$ and

(ii) the $2r \times r$ matrix $(H(\omega), H(\omega + \pi))$, $\omega \in \mathbb{C}$, has full rank, i.e.,

$$\operatorname{rank}(H(\omega), H(\omega + \pi)) = r, \qquad \omega \in \mathbb{C}.$$

In case $r = 1$ we always have $N(\phi) \neq \mathbb{C}$. Therefore the above condition (ii) is sufficient for $-1 \notin N(\phi)$.

3. POLYNOMIALS SPANNED BY THE SHIFTS OF REFINABLE VECTORS

For $p \in \Pi^r$, we will give conditions under which the semi-convolution $\phi *' p$ is a polynomial, where ϕ is a refinable vector. To this end, we need the following.

LEMMA 2. *Let $p \in \Pi^r$ be an eigenvector of S_a associated with eigenvalue c. For any integer $s \geqslant 0$, we define $F_s(\omega) = H(2^{s-1}\omega) \cdots H(\omega)\, F(\omega)$, where $F(\omega) = H(\omega/2)$. Then for any integer n it holds*

$$(p^{\mathrm{T}}(x - iD/2^s)\, F_s)(2n\pi) = c^s(p^{\mathrm{T}}(2^s x - iD)\, F)(2n\pi), \qquad x \in \mathbb{R}.$$

Proof. The proof proceeds with induction on s. For $s = 0$, there is nothing to prove. Assume that the claim is true for some $s-1$, $s \geqslant 1$. We set $H_{s-1}(\omega) = H(2^{s-1}\omega)$, so that $F_s = H_{s-1} F_{s-1}$. By the Leibnitz formula for differentiation we get

$$\begin{aligned}
&(p^{\mathrm{T}}(x/2 - iD/2^s)\, F_s)(2n\pi) \\
&\quad = \sum_{m \geqslant 0} (-i/2^s)^m\, (p^{(m)\mathrm{T}}(x/2 - iD/2^s)\, H_{s-1})(2n\pi)\, F_{s-1}^{(m)}(2n\pi)/m! \\
&\quad = \sum_{m \geqslant 0} (-i/2^s)^m \left(\sum_{k \in \mathbb{Z}} p^{(m)\mathrm{T}}\left(\frac{x-k}{2}\right) a_k \right) F_{s-1}^{(m)}(2n\pi)/m! \\
&\quad = c \sum_{m \geqslant 0} (-i/2^{s-1})^m\, p^{(m)\mathrm{T}}(x)\, F_{s-1}^{(m)}(2n\pi)/m! \\
&\quad = c \sum_{m \geqslant 0} p^{(m)\mathrm{T}}(x)(-iD/2^{s-1})^m\, F_{s-1})(2n\pi)/m! \\
&\quad = c(p^{\mathrm{T}}(x - iD/2^{s-1})\, F_{s-1})(2n\pi),
\end{aligned}$$

where in the third step we have used (iii) of Corollary 1. The proof is complete by the induction hypothesis. ∎

THEOREM 4. *Assume ϕ is a refinable vector associated with mask a. Let $p \in \Pi^r$ be an eigenvector of S_a. Then the following statements are true.*

(i) *The semi-convolution $\phi *' p$ is a polynomial.*

(ii) $(\phi *' p)^{(m)} = \phi *' p^{(m)}$.

(iii) *The degree of $\phi *' p$ is at most k when $p \in \Pi_k^r$.*

Proof. For any integer $j \neq 0$, write $j = 2^s n$, where $s \geqslant 0$ and n is an odd integer. Note $\hat{\phi}(2^s \omega) = F_s(\omega)\, \hat{\phi}(\omega)$, where F_s is given as in Lemma 2. Therefore we have

$$(p^{\mathrm{T}}(x - iD)\, \hat{\phi})(2j\pi) = (p^{\mathrm{T}}(x - iD/2^s)\, F_s \hat{\phi})(2n\pi).$$

Applying the Leibnitz formula to the right hand of above equation we obtain

$$\begin{aligned}(p^{\mathrm{T}}(x - iD)\, \hat{\phi})(2j\pi)& \\ &= \sum_{m \geqslant 0} (-i/2^s)^m\, (p^{(m)^{\mathrm{T}}}(x - iD/2^s)\, F_s)(2n\pi)\, \hat{\phi}^{(m)}(2n\pi)/m!.\end{aligned}$$

If p is an eigenvector of S_a associated with eigenvalue c, then for $m \geqslant 0$, $p^{(m)}$ is an eigenvector of S_a associated with $2^m c$. From Lemma 2 and (5) it follows that

$$(p^{(m)^{\mathrm{T}}}(x - iD/2^s)\, F_s)(2n\pi) = (2^m c)^s\, (p^{(m)^{\mathrm{T}}}(x - iD)\, F)(2n\pi) = 0, \qquad x \in \mathbb{R}.$$

Therefore we have

$$(p^{\mathrm{T}}(x - iD)\, \hat{\phi})(2j\pi) = 0, \qquad \forall j \neq 0, \qquad x \in \mathbb{R}. \tag{9}$$

Define the convolution of ϕ with the polynomial vector $p = (p_1, ..., p_r)^{\mathrm{T}}$ as

$$\phi * p = \sum_{1 \leqslant j \leqslant r} \phi_j * p_j,$$

where, as usual, $\phi_j * f_j$ is the convolution of a compactly supported distribution ϕ_j with a polynomial p_j. Clearly, $\phi * p \in \mathscr{S}'(\mathbb{R})$. Moreover we have the following Poisson's summation formula

$$\sum_{j \in \mathbb{Z}} p^{\mathrm{T}}(j)\, \phi(\cdot - j) = \sum_{j \in \mathbb{Z}} \phi * (p e_{2j\pi}),$$

where $e_{2j\pi}$ is the exponential function $x \mapsto e^{2j\pi x}$. The formula means that both sides converge to the same limit in the topology of distribution space $\mathscr{S}'(\mathbb{R})$. Using the above formula we obtain

$$\sum_{j \in \mathbb{Z}} p^{\mathrm{T}}(j)\, \phi(x - j) = \sum_{j \in \mathbb{Z}} (p^{\mathrm{T}}(x - iD)\, \hat{\phi})(2j\pi)\, e^{2jix\pi}.$$

By virtue of (9) we get

$$\phi *' p(x) = (p^{\mathrm{T}}(x - iD)\,\hat{\phi})(0) = \sum_{m \geqslant 0} p^{(m)^{\mathrm{T}}}(x)((-iD)^m\,\hat{\phi})(0)/m!,$$

which is a polynomial. This proves (i).

By the expression of $\phi *' p$ given as above, it is obvious that $(\phi *' p)^{(m)} = \phi *' p^{(m)}$. Moreover, for $p \in \Pi_k^r$, the degree of $\phi *' p$ is at most k, and is exactly equal to k if and only if $p^{(k)^{\mathrm{T}}}\hat{\phi}(0) \neq 0$. ∎

LEMMA 3. *If $x^k \in \mathscr{S}(\phi)$ then there exists a polynomial sequence q such that $x^k = \phi *' q$ and $q^{(k+1)} \in K(\phi)$.*

Proof. The existence of q was proved in [11]. It follows from (ii) of Theorem 4 immediately that $q^{(k+1)} \in K(\phi)$. ∎

We are in a position to characterize the accuracy of $\mathscr{S}(\phi)$ in terms of eigenvalues and eigenvectors in Π^r of the corresponding subdivision operator.

THEOREM 5. *Suppose that $-1 \notin N(\phi)$. Then $\mathscr{S}(\phi)$ has accuracy k if and only if there exists a polynomial sequence $p \notin K(\phi)$ such that p is an eigenvector of the subdivision S_a with eigenvalue 2^{-k+1}. If this is the case, then p has form $p = c\varepsilon_1 x^{k-1} + \sum_{m<k-1} p_m x^m$ provided that $H(0)$ has form (3), where $p_m \in \mathbb{C}^r$, and c is some nonzero constant.*

Proof. By Theorem JRZ one can prove the sufficiency of Theorem 5, even without the assumption that $-1 \notin N(\phi)$. We present another proof here due to its simplicity. Let $p \in \Pi^r \backslash K(\phi)$ such that $S_a p = 2^{-k+1} p$. It follows from (i) of Theorem 4 and equality (8) that the nonzero function $\phi *' p$ is a polynomial satisfying $\phi *' p(x) = 2^{-k+1} \phi *' p(2x)$. It implies that $\phi *' p$ is a nonzero multiple of x^{k-1}. This together with (ii) of Theorem 4 proves that $x^m \in \mathscr{S}(\phi)$, $m = 0, 1, ..., k-1$.

To prove the necessity, let q be given as in Lemma 3 such that $x^{k-1} = \phi *' q$. Hence $S_a q - 2^{-k+1} q \in K(\phi)$ by (8). In virtue of Theorem 3, $S_a q \in \Pi^r$. By Corollary 2, $K(\phi) \cap \Pi^r$ is an invariant subspace of S_a. Thus the linear subspace spanned by q and $K(\phi) \cap \Pi^r$ is also an invariant subspace of S_a with finite dimension. Note $q \in \Pi^r$ and $q \notin K(\phi)$. Therefore S_a has 2^{-k+1} as its eigenvalue and an associated eigenvector p such that $p \in \Pi^r$ and $p - q \in K(\phi)$. Consequently, we have

$$x^{k-1} = \phi *' q = \phi *' p. \tag{10}$$

Assume furthermore that $H(0)$ has form (3). It follows from $S_a p = 2^{-k+1}p$ and (i) of Theorem 2 that p has the form $p = c\varepsilon_1 x^{k-1} + \sum_{m<k-1} p_m x^m$. Since $p^{(k-1)} \neq 0$ by (10) and (ii) of Theorem 4, $c \neq 0$. The proof is complete. ∎

Remark 2. The main problem in the application of Theorem 5 is, for a given subdivision operator S_a, how to determine its eigenvalues of form 2^{-k} and the associated eigenvectors in Π^r. It was solved by Jia, Riemenschneider, and Zhou [11]. They reduced this problem to the similar one corresponding to a matrix of finite order, which will be explained in Section 4.

Although the following result is well known, we would like present it as a corollary of Theorem 5.

COROLLARY 3. *Assume* $r = 1$ *and* $\sum_j a(j) = 2$. *If the equality* (13) *holds then* $\mathcal{S}(\phi)$ *has accuracy k if and only if*

$$D^m H(\pi) = 0, \qquad 0 \leqslant m \leqslant k-1. \tag{11}$$

Proof. We first note that, under our hypotheses, $1 \notin N(\phi)$ so that $N(\phi) \neq \mathbb{C}$. Moreover by the sufficient conditions, which were stated at the end of Section 2, we have $-1 \notin K(\phi)$. By Theorem 5 $\mathcal{S}(\phi)$ has accuracy k if and only if there exists a polynomial p, $p = \sum_{m<k} p_m x^m$, $p_m \in \mathbb{C}$, and $p_{k-1} = 1$, satisfying that $S_a p = 2^{-k+1}p$ and, thus, $S_a p$ is a polynomial sequence. Therefore $S_a p^{(m)}$ is also a polynomial sequence for any $m \geqslant 0$. Since $\operatorname{span}\{x^m : 0 \leqslant m \leqslant k-1\} = \operatorname{span}\{p^{(m)} : m \geqslant 0\} = ST_p$, we have that $S_a x^m$, $m < k$, are all polynomial sequences. Therefore (11) follows from (5). ∎

The following example demonstrates that the condition "$-1 \notin N(\phi)$" cannot be removed neither in Theorem 3 nor in Theorem 5.

EXAMPLE. Let $r = 1$ and the refinement mask $a(z) = 1 + z^2$. The characteristic function of the interval $[0, 2]$, denoted by ϕ, is a solution to the refinement Eq. (1). It follows from the equality $\sum_{j \in \mathbb{Z}} \phi(\cdot - j) = 2$ that $\mathcal{S}(\phi)$ has accuracy 1. By definition we have $-1 \in N(\phi)$ and $1 \notin N(\phi)$. It is easy to check that there exists no polynomial sequence p such that $S_a p = p$.

Indeed if there is such a p, by (ii) of Corollary 1, $p \in \Pi_0$. Therefore, without loss of generality, we assume that $p = 1$. However, $H(\pi)\, p \neq 0$, which is a contradiction with (i) of Corollary 1. Of course we have $S_a p - p = 2((-1)^j)_{j \in \mathbb{Z}} \in K(\phi)$ either by Theorem JRZ or by a direct observation.

4. NECESSARY CONDITIONS FOR SMOOTHNESS OF REFINABLE VECTORS

In this section we give some necessary conditions for the refinable vectors to be smooth. The conditions are expressed in terms of the nondegeneraty and simplicity of eigenvalues of some matrices. For $r=1$, the existence of these eigenvalues has been obtained by many authors. Our proofs are very simple and depend on a technique of extension given in [11].

For a finitely supported mask a, without loss of generality we assume that there is an integer $N>0$ such that $a(j)=0$ for $j \notin [-N, N]$. In this case the support of the refinable vector ϕ is contained in $[-N, N]$ (cf. [3]). Consequently, if all components of ϕ are continuous on $\mathbb{R}$, $\phi(x)=0$, for $|x| \geqslant N$. Moreover if ϕ is not identically zero and the kth derivatives of the components of ϕ are continuous, then vector $(\phi^{(m)}(j))_{|j| \leqslant N} \neq 0$, $m=0, 1, ..., k$.

Let Λ be a finitely dimensional subspace given by $\Lambda := \{\lambda : \lambda = (\lambda(j)), \lambda(j) \in \mathbb{C}^r, |j| \leqslant N\}$. We use the same notation λ to denote a vector in $\lambda \in \Lambda$ and its extension $\lambda \in (l(\mathbb{Z}))^r$. The operator S defined on Λ is given by

$$S\lambda(j) = \sum_{|k| \leqslant N} a^{\mathrm{T}}(j-2k)\, \lambda(k), \qquad j \in [-N, N].$$

The above operator can be also regarded as a matrix. Note that, for fixed $j>N$, the support of sequence $(a(j-2k))_{k \in \mathbb{Z}}$ is contained in $\{1, 2, ..., j-1\}$; and for fixed $j<-N$, the support of $(a(j-2k))_{k \in \mathbb{Z}}$ is contained in $\{j+1, j+2, ..., -1\}$. Therefore $S\lambda(j) = S_a \lambda(j)$ for $|j| \leqslant N$.

Evidently, any eigenvalue of S_a is also an eigenvalue of S, and for the eigenvector $\lambda = (\lambda(j))_{j \in \mathbb{Z}} \in (l(\mathbb{Z}))^r$ of S_a, the restriction $(\lambda(j))_{|j| \leqslant N} \in \Lambda$ is an eigenvector of S. More importantly, the partial converse is true. It is proved in [11] that any nonzero eigenvalue of S is one of S_a and the associated eigenvector of S can be extended to an eigenvector of S_a. Furthermore, if the eigenvector $\lambda \in \Lambda$ satisfies $\lambda(j) = p(j)$ for $|j| \leqslant N$ and some $p \in \Pi^r$, then $(p(j))_{j \in \mathbb{Z}}$ is just the extended vector. Therefore we can obtain all nonzero eigenvalues and eigenvectors in Π^r of S_a in this way. We present the method of extension in [11] for convenience.

Let $\lambda \in \Lambda \backslash \{0\}$ satisfy $S\lambda = c\lambda$ with $c \neq 0$. We can define $\lambda(j)$ for $j>N$ recursively by

$$\lambda(j) = \frac{1}{c} \sum_{k=1}^{j-1} a^{\mathrm{T}}(j-2k)\, \lambda(k). \tag{12}$$

Similarly, if $j < -N$, let

$$\lambda(j) = \frac{1}{c} \sum_{k=j+1}^{-1} a^{\mathrm{T}}(j-2k)\, \lambda(k). \tag{13}$$

By the support property of sequence $(a(j-2k))_{k \in \mathbb{Z}}$, given as above, for $j > N$ or $j < -N$ fixed, we observe that the extended vector λ, as given by (12) and (13), satisfies $S_a \lambda = c\lambda$.

The eigenvalue c of the matrix S is said to be *degenerate*, if there exist nonzero vectors λ and λ_1 satisfying $S\lambda = c\lambda$ and $S\lambda_1 = \lambda + c\lambda_1$. For $c \neq 0$, using the above method we may extend λ_1 as well as λ to vectors in $(l(\mathbb{Z}))^r$ such that

$$S_a \lambda = c\lambda \qquad \text{and} \qquad S_a \lambda_1 = \lambda + c\lambda_1. \tag{14}$$

Indeed, besides (12) and (13) we should set

$$\lambda_1(j) = \frac{1}{c} \sum_{k=1}^{j-1} a^{\mathrm{T}}(j-2k)\, \lambda_1(k) + \lambda(j), \qquad j > N$$

and

$$\lambda_1(j) = \frac{1}{c} \sum_{k=j+1}^{-1} a^{\mathrm{T}}(j-2k)\, \lambda_1(k) + \lambda(j), \qquad j < -N.$$

LEMMA 4. *Assume that the kth derivatives of the components of the refinable vector ϕ are continuous on $\mathbb{R}$. Let $c \neq 0$ be an eigenvalue of S_a and λ the associated eigenvector. Then the function $f_\lambda := \phi *' \lambda$ satisfies the following conditions*:

(i) $f_\lambda^{(k)}(0) = 0$ *if* $c \neq 1/2^k$;

(ii) $f_\lambda^{(k)}(x) = f_\lambda^{(k)}(0)$, $x \in \mathbb{R}$ *if* $c = 1/2^k$;

(iii) $f_\lambda^{(k)}(x) = 0$, $x \in \mathbb{R}$, *if* $c \neq 1/2^k$ *and* $|c| \geqslant 1/2^k$.

Proof. We observe that for the function f_λ given above satisfies $f_\lambda(x) = c f_\lambda(2x)$ by (8). Therefore for any integer $n \geqslant 1$ it holds

$$f_\lambda^{(k)}(x) = (2^k c)^{-n} f_\lambda^{(k)}(2^{-n} x), \qquad x \in \mathbb{R}. \tag{15}$$

So, (i) follows by setting $x = 0$ in (15). Furthermore, letting $n \to \infty$ in (15) we obtain (ii).

If $|c| \geqslant 1/2^k$ it follows from (15) that $|f_\lambda^{(k)}(x)| \leqslant |f_\lambda^{(k)}(2^{-n}x)|$, $x \in \mathbb{R}$, which gives $|f_\lambda^{(k)}(x)| \leqslant |f_\lambda^{(k)}(0)|$, $x \in \mathbb{R}$. The proof of (iii) is complete by (i). ∎

THEOREM 6. *Assume that the kth derivatives of the components of the refinable vector ϕ are continuous on $\mathbb{R}$. The following statements are true.*

(i) $\mathcal{S}(\phi)$ *has accuracy* $k+1$.

(ii) $1, 1/2, ..., 1/2^k$ *are nondegenerate eigenvalues of S.*

(iii) *If the shifts of ϕ are linearly independent, the characteristic polynomial of matrix S has* $1, 1/2, ..., 1/2^k$ *as its simple zeros, and has no other zero c with* $|c| \geqslant 1/2^k$.

Proof. For simplicity we only present the required arguments for $m=k$. The reasoning is the same for $m<k$. Define

$$\Lambda_\sigma = \left\{ \lambda\colon \lambda \in \Lambda,\ \sum_{|j| \leqslant N} \lambda^{\mathrm{T}}(j)\, \phi^{(k)}(-j) = \sigma \right\}, \qquad \sigma = 0, 1.$$

Since vector $(\phi^{(k)}(j))_{|j| \leqslant N} \neq 0$, Λ_1 is not empty. Choosing an element, say, μ_1, from Λ_1, we may write any vector $\lambda \in \Lambda$ as $\lambda = b\mu_1 + \mu_0$, where $b = \sum_{|j| \leqslant N} \lambda^{\mathrm{T}}(j)\, \phi^{(k)}(-j)$ and $\mu_0 \in \Lambda_0$. To get such a representation for $S\lambda$, we differentiate equality (8) k times and envalue at $x=0$. Note that, for $x=0$, the nonzero summands in the sums of (8) are in Λ. It follows that

$$\sum_{|j| \leqslant N} \lambda^{\mathrm{T}}(j)\, \phi^{(k)}(-j) = 2^k \sum_{|j| \leqslant N} (S\lambda)^{\mathrm{T}}\,(j)\, \phi^{(k)}(-j).$$

Therefore Λ_0 is an invariant subspace of S and the representation of $S\lambda$ is

$$S\lambda = 1/2^k b\mu_1 + S\mu_0, \qquad S\mu_0 \in \Lambda_0. \tag{16}$$

It follows from (16) that S has an eigenvector $\lambda \in \Lambda_1$ with eigenvalue $1/2^k$. The extended vector $\lambda \in (l(\mathbb{Z}))^r$ satisfies $S_a \lambda = 1/2^k \lambda$ and defines a function $f_\lambda = \phi *' \lambda$. Note that, by (i) of Lemma 4, $f_\lambda^{(m)}(0) = 0$ for $m = 0, 1, ..., k-1$. On the other hand, $\lambda \in \Lambda_1$ and (iii) of Lemma 4 give $f_\lambda^{(k)}(x) = f_\lambda^{(k)}(0) = 1$. We thus have $f_\lambda(x) = x^k/k! \in \mathcal{S}(\phi)$.

If $1/2^k$ is degenerate we can find a λ_1 such that the equalities in (14) hold with $c = 1/2^k$. Therefore, by (8), $f_{\lambda_1}(x) = (2x)^k/k! + 2^{-k} f_{\lambda_1}(2x)$. Differentiating it k times and envaluing at $x=0$ yields $2^k = 0$, a contradiction.

Assume that $1/2^k$ is not simple. By (ii), associated with eigenvalue $1/2^k$ of S, besides the eigenvector λ given above, there exists another eigenvector μ such that λ and μ are linearly independent. Without loss of generality we may assume that $\mu \in \Lambda_0$. Thus the function $f_\mu := \phi *' \mu$ becomes zero by (i) and (ii) of Lemma 4. It contradicts the assumption that the shifts of ϕ are linearly independent.

Secondly let c be an eigenvalue of S with $|c| \geqslant 1/2^k$, $c \neq 1, 1/2, ..., 1/2^k$ and v the corresponding eigenvector. Thus we have a function $f_v := \phi *' v$ such that $f_v(x) = c^{-n} f_v(2^{-n}x)$. Since $c \neq 1, 1/2, ..., 1/2^{k-1}$, $f_v^{(m)}(0) = 0$, $m = 0, ..., k-1$, by (i) of Lemma 1. Moreover $c \neq 1/2^k$ yields $f_v^{(k)}(x) = 0$. Therefore $f_v(x) = 0$, again a contradiction. ∎

Remark 3. Under a weaker condition, the analogy of (i) of Theorem 6 for $r = 1$ and multivariate case have been proved in [7] by a different method.

At the end of the paper let us take a quick look at a necessary condition for $\phi^{(k)} \in L_2$ under the assumption that the shifts of ϕ are linearly independent. Let $\psi = \phi * \phi^*$ with $\phi^* = \overline{\phi(-\,\cdot)}$. Observe that ψ is a refinable function with refinement mask $b = (b(j))_{j \in \mathbb{Z}}$, where $b(j) = \frac{1}{2} \sum_{m \in \mathbb{Z}} a(m) \overline{a(j+m)}$. Moreover, $\psi^{(2k)}$ is continuous on $\mathbb{R}$ when $\phi^{(k)} \in L_2$. In this case we may apply Theorem 6 to the operator S determined by mask b. Therefore it has nondegenerate eigenvalues $1, 1/2, ..., 1/2^{2k}$. Moreover, from Lemma 3 and the proof of (i) of Theorem 6 we know that S_b has eigenvector of polynomial sequence $p \in \Pi$ with eigenvalues $1, 1/2, ..., 1/2^{2k}$.

For a compactly supported distribution ϕ, the shifts of ϕ are linearly independent if and only if for any $z \in \mathbb{C}$, the vector $(\hat{\phi}(z + 2k\pi))_{k \in \mathbb{Z}} \neq 0$. It is not difficult to prove that if ϕ has linearly independent shifts, so does ψ. We conclude that, if the shifts of ϕ are linearly independent and $\phi^{(k)} \in L_2$, the characteristic polynomial of matrix S associated with b has the numbers $1, 1/2, ..., 1/2^{2k}$ as its simple zeros and has no other zero c with $|c| \geqslant 1/2^{2k}$.

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